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Large Deviations for Noninteracting Infinite-Particle Systems

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A large deviation property is established for noninteracting infinite particle systems. Previous large deviation results obtained by the authors involved a single I-function because the cases treated always involved a unique invariant measure for the process. In the context of this paper there is an infinite family of invariant measures and a corresponding infinite family of I-functions governing the large deviations.

KEY WORDS: Large deviations; infinite particle systems; invariant measures; asymptotics for expectations.

1. INTRODUCTION

Let X be a countable set and let $X_0, X_1, X_2,...$ be a Markov chain with state space X and transition probabilities $\{\pi_{xy}\}$. Probabilities and expectations for this chain starting from a point $x \in X$ will be denoted by $P_x^\pi\{\ \}$ and $E_x^\pi\{\ \}$, respectively. We make five assumptions about this chain: it is irreducible, transient, the matrix $\{\pi_{xy}\}$ is doubly stochastic, and if for any finite set $F \subset X$ we define $\tau_F = \inf_{j \geq 1}\{X_j \in F\}$, then we assume $\lim_{x \to \infty} P_x^\pi\{\tau_F < \infty\} = 0$, i.e., for any $\varepsilon > 0$ there exists another finite set F_1 such that $F \subset F_1$ and $x \notin F_1$ implies $P_x^\pi\{\tau_F < \infty\} < \varepsilon$. Finally, we assume that the Green's function $G(x, y)$ for the chain has the property $\sup_{x \in X} G(x, x) < \infty$.

For each $x \in X$, let $n_0(x)$ be a nonnegative integer giving the number of particles at x initially, so that $\{n_0(x), x \in X\}$ is the initial configuration of particles. At time 1 all of these particles move independently according to the transition probabilities $\{\pi_{xy}\}$, giving us a new configuration

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$\{n_1(x), x \in X\}$. There is no interaction here; all the $n_0(x)$ particles at x leave independently of one another and for any other point $y \in X$, the $n_0(y)$ particles at y leave independently not only of each other, but of all the particles at x. At time 2 the process is repeated, giving us a new configuration $\{n_2(x), x \in X\}$ and so on. If we let $N = \{0, 1, 2,...\}$, then $Z = N^X$ is the space of all configurations, and if at time j we have a particular configuration $n(\cdot) \in Z$, we can ask for the transition probability $\hat{\pi}(n(\cdot), n'(\cdot))$ that at time $j+1$ we have configuration $n'(\cdot) \in Z$.

From our assumptions, the transition probability $\hat{\pi}(n(\cdot), n'(\cdot))$ is given through its moment generating function by

$$\int_Z \left\{\exp\left[-\sum_{x \in X} \lambda(x)\, n'(x)\right]\right\} \hat{\pi}(n(\cdot), dn'(\cdot)) = \prod_{x \in X} \left(\sum_{y \in X} \{\exp[-\lambda(y)]\}\, \pi_{xy}\right)^{n(x)} \tag{1.1}$$

where $\lambda: X \to \mathbb{R}$ vanishes outside some finite subset of X.

Important for us will be the family $\underline{A}$ of σ-finite invariant measures $\alpha(\cdot)$ for π_{xy}, i.e., for all $x \in X$, $0 \leqslant \alpha(x) < \infty$, $\sum_{x \in X} \alpha(x)\, \pi_{xy} = \alpha(y)$, and we assume $\alpha(\cdot) \not\equiv 0$. Since the chain is irreducible, if $\alpha(\cdot)$ vanishes at any point, it vanishes everywhere, so actually $\alpha(x) > 0$ for all $x \in X$.

For each $\alpha \in \underline{A}$ we define a measure P_α on Z as follows: for each $x \in X$ the number of particles at x, i.e., $n(x)$, is Poisson-distributed with mean $\alpha(x)$ and $\{n(x), x \in X\}$ are mutually independent.

It is well known and, using (1.1), easy to show that for each $\alpha \in \underline{A}$, P_α is an invariant measure for $\hat{\pi}(\cdot, \cdot)$. Also, using the Kolmogorov zero-one law, we can show that for each $\alpha \in \underline{A}$, P_α is an ergodic measure on Z.

For each $x \in X$ and any positive integer N, let

$$L_N(x) = \frac{n_0(x) + n_1(x) + \cdots + n_{N-1}(x)}{N}$$

i.e., the average number of particles at site x during the first N steps. Let Σ be the space of sequences of particle configurations on X, i.e., $\{n_j(\cdot), j = 0, 1, 2,...\}$. Any initial configuration $n(\cdot)$ of particles and $\hat{\pi}(\cdot, \cdot)$ generate a probability measure on Σ which we will denote by $\underline{P}_n$ [we will often in the sequel use $n(\cdot)$ instead of $n_0(\cdot)$ as a generic notation for the initial configuration]. Let $\underline{M}$ be the space of σ-finite measures σ on X. Then, for each N, L_N maps Σ into $\underline{M}$, and we use this mapping to define a probability measure $Q_{n,N}$ on $\underline{M}$ by $Q_{n,N} = \underline{P}_n \cdot L_N^{-1}$, i.e., if $A \subset \underline{M}$, then $Q_{n,N}(A) = \underline{P}_n\{L_N \in A\}$.

Since, for each $\alpha \in \underline{A}$, P_α is an invariant measure for $\hat{\pi}(\cdot, \cdot)$ and each P_α

is ergodic, it follows from the ergodic theorem that as $N \to \infty$, $Q_{n,N} \Rightarrow \delta_\alpha$ where, of course, δ_α is the Dirac measure on $\underline{M}$ concentrated at $\alpha(\cdot)$. What is essential is that the statement $Q_{n,N} \Rightarrow \delta_\alpha$ holds *for almost all* $n(\cdot) \in Z$ *with respect to* P_α*-measure.* This makes sense since if $\alpha_1 \not\equiv \alpha_2$, then P_{α_1} and P_{α_2} are mutually singular measures on Z.

In this paper we prove that a large-deviation property holds for the $Q_{n,N}$ measure, i.e., we find a functional $I_\alpha(\sigma)$ that for each $\alpha \in \underline{A}$ maps $\underline{M}$ into $[0, \infty]$ such that for each closed set $C \subset \underline{M}$,

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \log Q_{n,N}(C) \leqslant -\inf_{\sigma \in C} I_\alpha(\sigma) \tag{1.2}$$

and for each open set $G \subset \underline{M}$,

$$\underline{\lim_{N \to \infty}} \frac{1}{N} \log Q_{n,N}(G) \geqslant -\inf_{\sigma \in G} I_\alpha(\sigma) \tag{1.3}$$

where both of the statements (1.2) and (1.3) hold for almost all $n(\cdot)$ with respect to P_α-measure on Z.

This last "almost all" statement emphasizes what is new in this paper relative to our previous work (see Refs. 1–3 for the theory, and Refs. 4–6 for some applications), namely that we have here more than one invariant measure. In our earlier papers we always imposed sufficient hypotheses on the underlying process so that there was a unique invariant measure and hence a single I-function governing large deviations. Here the underlying process has a whole family of invariant measures $\{P_\alpha, \alpha \in \underline{A}\}$, and we have an I-function therefore corresponding to each $\alpha \in \underline{A}$.

We obtain an explicit expression for $I_\alpha(\sigma)$, which we discuss now. Let $\underline{V}$ be the space of functions $V: X \to \mathbb{R}$ that vanish outside some finite subset of X and have the property that for some $x \in X$,

$$u_V(x) = u(x) = E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} < \infty \tag{1.4}$$

It is an elementary consequence of the irreducibility hypothesis (see first few lines of Lemma 2.1 below) that $u(x) < \infty$ for all $x \in X$ if it is finite for any x.

Now, for $\alpha \in \underline{A}$ and $\sigma \in \underline{M}$, define

$$I_\alpha(\sigma) = \sup_{V \in \underline{V}} \left\{ \sum_{x \in X} \sigma(x)\, V(x) - \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u(x) \right\} \tag{1.5}$$

and this is the I-function that enters into (1.2) and (1.3). For a given $\alpha \in \underline{A}$, $I_\alpha(\sigma)$ turns out to be finite if and only if σ behaves "asymptotically" like α. This is made explicit in Theorem 3.3 below. If $\alpha \in \underline{A}$ and $\beta \in \underline{A}$ and $\sigma \in \underline{M}$ exists such that both $I_\alpha(\sigma) < \infty$ and $I_\beta(\sigma) < \infty$, then $\alpha \equiv \beta$ (Lemma 3.4). In

Section 3 other properties of this I-function are established. In Section 2 we prove the upper bound (1.2) and in Section 4 we prove the lower bound (1.3).

Once such a large-deviation property for $Q_{n,N}$ is established, i.e., once one has an $I_\alpha(\sigma)$ for which (1.2) and (1.3) hold, then one easily obtains[7] an asymptotic result for certain expectations with respect to the $\underline{P}_n$ measure. To make this explicit, let Φ: $\underline{M} \to \mathbb{R}$ be bounded and continuous; then (1.2) and (1.3) imply

$$\lim_{N\to\infty} \frac{1}{N} \log E^{\underline{P}_n}\{e^{N\Phi(L_N(\cdot))}\} = \sup_{\sigma\in\underline{M}} [\Phi(\sigma) - I_\alpha(\sigma)] \tag{1.6}$$

for almost all $n(\cdot) \in Z$ (P_α-measure).

To obtain large-deviation properties where more than one invariant measure is involved, the present context of infinite-particle systems is natural. Here, in the simplest situation of noninteracting particles, we obtain an explicit I-function (1.5) in terms of which (1.2) and (1.3) are proved. In the case of infinite-particle systems with interaction the problem is to find the appropriate I-function in terms of which the analogues of (1.2) and (1.3) can be established.

Some interesting large-deviation results for infinite-particle systems with interaction have been found by Cox and Griffeath.[9] Lee[8] obtained a large-deviation property with explicit I-function for an infinite-particle system with no interaction, but where the particles move according to independent Brownian motions with constant drift.

2. THE UPPER BOUND

Using the definitions and notation of the introduction, let $V \in \underline{V}$ and C be a closed subset of $\underline{M}$. Then,

$$\begin{aligned} &E^{\underline{P}_n}\left\{\exp\left[\sum_{j=0}^{N-1} \sum_{x\in X} n_j(x)\, V(x)\right]\right\} \\ &= E^{\underline{P}_n}\left\{\exp\left[N \sum_{x\in X} L_N(x)\, V(x)\right]\right\} \\ &= E^{Q_{n,N}}\left\{\exp\left[N \sum_{x\in X} \sigma(x)\, V(x)\right]\right\} \\ &\geqslant E^{Q_{n,N}}\left\{\exp\left[N \sum_{x\in X} \sigma(x)\, V(x)\right]; \sigma\in C\right\} \\ &\geqslant \left\{\exp\left[N \inf_{\sigma\in C} \sum_{x\in X} \sigma(x)\, V(x)\right]\right\} Q_{n,N}(C) \end{aligned} \tag{2.1}$$

In Theorem 2.10 below we show

$$\lim_{N\to\infty} \frac{1}{N} \log E^{P_n}\left\{\exp\left[\sum_{j=0}^{N-1} \sum_{x\in X} n_j(x)\, V(x)\right]\right\}$$
$$= \sum_{x\in X} \alpha(x)(1-e^{-V(x)})\, u(x) \tag{2.2}$$

for almost all $n(\cdot)\in Z$ (P_α-measure). We recall from the introduction that

$$u(x) = E_x^{\pi}\left\{\exp\left[\sum_{j=0}^{\infty} V(X_j)\right]\right\}$$

and that if $\alpha_1 \not\equiv \alpha_2$, then P_{α_1} and P_{α_2} are mutually singular measures on Z.

Thus, from (2.1) and (2.2) we conclude

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{n,N}(C)$$
$$\leqslant -\inf_{\sigma\in C} \sum_{x\in X} \sigma(x)\, V(x) + \sum_{x\in X} \alpha(x)(1-e^{-V(x)})\, u(x)$$
$$= -\inf_{\sigma\in C}\left[\sum_{x\in X} \sigma(x)\, V(x) - \sum_{x\in X} \alpha(x)(1-e^{-V(x)})\, u(x)\right] \tag{2.3}$$

for almost all $n(\cdot)\in Z$ (P_α-measure). Since (2.3) holds for any $V\in \underline{V}$, we obtain

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{n,N}(C)$$
$$\leqslant -\sup_{V\in \underline{V}} \inf_{\sigma\in C}\left[\sum_{x\in X} \sigma(x)\, V(x) - \sum_{x\in X} \alpha(x)(1-e^{-V(x)})\, u(x)\right] \tag{2.4}$$

for almost all $n(\cdot)\in Z$ (P_α-measure). In Lemma 2.11 below we show that, because C is closed, (2.4) implies

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{n,N}(C)$$
$$\leqslant -\inf_{\sigma\in C} \sup_{V\in \underline{V}}\left[\sum_{x\in X} \sigma(x)\, V(x) - \sum_{x\in X} \alpha(x)(1-e^{-V(x)})\, u(x)\right]$$
$$= -\inf_{\sigma\in C} I_\alpha(\sigma) \tag{2.5}$$

for almost all $n(\cdot)\in Z$ (P_α-measure).

Now (2.5) is just the upper bound (1.2) of the introduction, and hence what needs to be proved in this section is (2.2) and Lemma 2.11. We prove (2.2) as Theorem 2.10 after a succession of preliminary lemmas.

Let $\underline{V}_1$ be the space of functions $V: X \to \mathbb{R}$ such that V vanishes outside some finite subset of X and

$$u(x) = E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} < \infty$$

for *all* $x \in X$.

For F a finite subset of X, let $\tau = \tau_F = \inf_{j \geqslant 1}\{X_j \in F\}$, i.e., the first entry time of the Markov chain into F, and let $\pi_{xy}^F = P_x^\pi\{X_\tau = y,\ \tau < \infty\}$. Let M_F be the matrix $\{e^{V(x)}\pi_{xy}^F,\ x, y \in F\}$, where V vanishes outside F, and let $\rho(M_F)$ be the spectral radius of M_F. Define $\underline{V}_2$ to be the space of functions $V: X \to \mathbb{R}$ such that V vanishes outside some finite set $F = F_V$ and $\rho(M_F) < 1$.

Lemma 2.1. $\underline{V} = \underline{V}_1 = \underline{V}_2$.

Proof. Let $V \in \underline{V}$ and $x \in X$ such that

$$u(x) = E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} < \infty$$

Since the chain is irreducible, for any $y \in X$ there is a positive integer k such that $\pi_{xy}^{(k)} > 0$. If we let

$$\|V\| = \sup_{z \in F = \operatorname{supp} V} |V(z)|$$

then

$$\begin{aligned}
\infty &> E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} \\
&\geqslant [\exp(-k\|V\|)]\, E_x^\pi \left\{ \exp\left[\sum_{j=k}^{\infty} V(X_j) \right] \right\} \\
&\geqslant [\exp(-k\|V\|)]\, E_x^\pi \left\{ \exp\left[\sum_{j=k}^{\infty} V(X_j) \right]; X_k = y \right\} \\
&= [\exp(-k\|V\|)]\, \pi_{xy}^{(k)} E_y^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\}
\end{aligned}$$

so that

$$E_y^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} < \infty$$

which means $V \in \underline{V}_1$ and hence $\underline{V} = \underline{V}_1$.

To show $\underline{V}_1 \subset \underline{V}_2$, let $V \in \underline{V}_1$, so that

$$u(x) = E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} < \infty$$

for all $x \in X$. Let $F \subset X$ be the finite set outside of which V vanishes, and with τ and π_{xy}^F as defined just before this lemma, let

$$\eta_x = P_x^\pi\{\tau = \infty\} = 1 - \sum_{y \in F} \pi_{xy}^F$$

We note that, since the chain is transient, $\eta_x > 0$ for at least one $x \in F$.

Now, for any $x \in F$, using the strong Markov property of the chain with respect to τ,

$$\begin{aligned}
u(x) &= E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} \\
&= [\exp V(x)]\, E_x^\pi \left\{ \exp\left[\sum_{j=1}^{\infty} V(X_j) \right]; \tau < \infty \right\} \\
&\quad + [\exp V(x)]\, E_x^\pi \left\{ \exp\left[\sum_{j=1}^{\infty} V(X_j) \right]; \tau = \infty \right\} \\
&= [\exp V(x)]\, E_x^\pi \left\{ \exp\left[\sum_{j=1}^{\infty} V(X_j) \right]; \tau < \infty \right\} \\
&\quad + [\exp V(x)]\, P_x^\pi\{\tau = \infty\} \\
&= [\exp V(x)] \sum_{y \in F} \pi_{xy}^F E_y^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} + [\exp V(x)]\, \eta_x \\
&= [\exp V(x)] \sum_{y \in F} \pi_{xy}^F u(y) + [\exp V(x)]\, \eta_x
\end{aligned} \tag{2.6}$$

Let u be the column vector with entries $\{u(x),\ x \in F\}$, $e^V \eta$ be the column vector with entries $\{e^{V(x)} \eta_x,\ x \in F\}$; then, with M_F the matrix defined above, we can rewrite (2.6) as

$$(I - M_F)\, u = e^V \eta \tag{2.7}$$

Since (2.7) has the strictly positive solution

$$u(x) = E_x^\pi \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\}, \qquad x \in F$$

we conclude that $\rho(M_F) < 1$, i.e., $V \in \underline{V}_2$.

Finally, to show $\underline{V}_2 \subset \underline{V}_1$, let $V \in \underline{V}_2$ with finite support F. Since we are now assuming $\rho(M_F) < 1$ and, as noted earlier, at least one entry in $e^V \eta$ is strictly positive, the matrix equation $(I - M_F)\, u = e^V \eta$ for u has a solution $\bar{u}(x)$, $x \in F$. Indeed,

$$\bar{u} = (I - M_F)^{-1}(e^V \eta) \tag{2.8}$$

and since the chain is irreducible, $\bar{u} = \{\bar{u}(x), x \in F\}$ has strictly positive entries. We extend the column vector $\bar{u}$ by defining for $y \notin F$,

$$\begin{aligned}\bar{u}(y) &= \sum_{z \in F} \pi^F_{yz} \bar{u}(z) + P^\pi_y\{\tau = \infty\} \\ &= \sum_{z \in F} \pi^F_{yz} \bar{u}(z) + 1 - \sum_{z \in F} \pi^F_{yz} \end{aligned} \tag{2.9}$$

so that now $\bar{u}$ is a column vector having entries for all the elements of X. Thus, for any $x \in X$,

$$\begin{aligned}\sum_{y \in X} \pi_{xy} \bar{u}(y) &= \sum_{y \in F} \pi_{xy} \bar{u}(y) + \sum_{y \notin F} \pi_{xy} \bar{u}(y) \\ &= \sum_{y \in F} \pi_{xy} \bar{u}(y) + \sum_{y \notin F} \pi_{xy} \sum_{z \in F} \pi^F_{yz} \bar{u}(z) + \sum_{y \notin F} \pi_{xy} P^\pi_y\{\tau = \infty\} \end{aligned} \tag{2.10}$$

But,

$$\begin{aligned}\sum_{z \in F} \pi^F_{xz} \bar{u}(z) &= E^\pi_x\{\bar{u}(X_\tau); \tau < \infty\} \\ &= E^\pi_x\{\bar{u}(X_\tau); \tau = 1\} + E^\pi_x\{\bar{u}(X_\tau); 2 \leqslant \tau < \infty\} \\ &= \sum_{y \in F} \pi_{xy} \bar{u}(y) + \sum_{y \notin F} \pi_{xy} \sum_{z \in F} \pi^F_{yz} \bar{u}(z)\end{aligned}$$

and using this in (2.10), we have, for any $x \in X$,

$$\begin{aligned}\sum_{y \in X} \pi_{xy} \bar{u}(y) &= \sum_{z \in F} \pi^F_{xz} \bar{u}(z) + \sum_{y \notin F} \pi_{xy} P^\pi_y\{\tau = \infty\} \\ &= \sum_{z \in F} \pi^F_{xz} \bar{u}(z) + P^\pi_x\{\tau = \infty\}\end{aligned} \tag{2.11}$$

First consider (2.11) for $x \notin F$. Using (2.9), we obtain

$$\sum_{y \in X} \pi_{xy} \bar{u}(y) = \bar{u}(x)$$

On the other hand, if $x \in F$, then (2.11) and (2.8) state that for $V \in \underline{V}_2$,

$$\begin{aligned} e^{V(x)} & \sum_{y \in X} \pi_{xy} \bar{u}(y) \\ &= e^{V(x)} \sum_{z \in F} \pi^F_{xy} \bar{u}(z) + e^{V(x)} P^\pi_x\{\tau = \infty\} \\ &= e^{V(x)} \sum_{z \in F} \pi^F_{xz} \bar{u}(z) + e^{V(x)} \eta_x \\ &= (M_F \bar{u})_x + (e^V \eta)_x = \bar{u}(x) \end{aligned}$$

Since V vanishes outside of F, we conclude that for all $x \in X$,

$$e^{V(x)} \sum_{y \in X} \pi_{xy} \bar{u}(y) = \bar{u}(x)$$

or, with the abbreviated notation $(\pi\bar{u})(x) = \sum_{y \in X} \pi_{xy} \bar{u}(y)$, we have for all $x \in X$

$$e^{V(x)}(\pi\bar{u})(x) = \bar{u}(x) \tag{2.12}$$

Identity (2.12) holds for a $V \in \underline{V}_2$ and $\bar{u}$ defined by (2.8) and (2.9). In this connection we should note that for $V \in \underline{V}$ the same identity holds for

$$\begin{aligned} u(x) &= E^\pi_x \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} \\ &= [\exp V(x)]\, E^\pi_x \left\{ \exp\left[\sum_{j=1}^{\infty} V(X_j) \right] \right\} \\ &= [\exp V(x)] \sum_{y \in X} \pi_{xy} E^\pi_y \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\} \\ &= [\exp V(x)](\pi u)(x) \end{aligned} \tag{2.13}$$

Since $\bar{u}(x)$ is strictly positive for $x \in F$, we have

$$0 < \inf_{x \in F} \bar{u}(x) \leqslant \sup_{x \in F} \bar{u}(x) < \infty$$

From this and (2.9) we see that for all $y \in X$

$$0 < \min[1, \inf_{z \in F} \bar{u}(z)] \leqslant \bar{u}(y) \leqslant \max[\sup_{z \in F} \bar{u}(z), 1] < \infty \tag{2.14}$$

Now, (2.12) implies that the sequence of random variables

$$Z_n = \bar{u}(X_n) \exp\left[\sum_{j=0}^{n-1} V(X_j) \right]$$

is a martingale because if $\underline{F}_{n-1}$ is the σ-field up to time $n-1$,

$$E_x^\pi\{Z_n \mid \underline{F}_{n-1}\} = \left\{\exp\left[\sum_{j=0}^{n-1} V(X_j)\right]\right\} (\pi\bar{u})(X_{n-1})$$

$$= \left\{\exp\left[\sum_{j=0}^{n-2} V(X_j)\right]\right\} \bar{u}(X_{n-1}) = Z_{n-1}$$

Thus, for all $x \in X$,

$$E_x^\pi\left\{\left[\exp\sum_{j=0}^{n-1} V(X_j)\right]\bar{u}(X_n)\right\} = \bar{u}(x) \tag{2.15}$$

and from this and (2.14) we conclude, for all $x \in X$,

$$E_x^\pi\left\{\exp\left[\sum_{j=0}^{n-1} V(X_j)\right]\right\} \leqslant \frac{\bar{u}(x)}{\inf_{y\in X} \bar{u}(y)} < \infty$$

Finally, from Fatou's lemma, this last inequality implies

$$u(x) = E_x^\pi\left\{\exp\left[\sum_{j=0}^{\infty} V(X_j)\right]\right\} < \infty$$

for all $x \in X$, i.e., $V \in \underline{V}_1$, and the lemma is proved.

Lemma 2.2. Let F be a finite subset of X and let $\phi(x) = P_x^\pi\{\inf_{j\geqslant 1}(X_j \in F) < \infty\}$. Then for all $x \in X$, $\lim_{n\to\infty} E_x^\pi\{\phi(X_n)\} = 0$.

Proof. This is obvious, since

$$E_x^\pi\{\phi(X_n)\} = P_x^\pi\{X_j \in F, j > n\}$$

$$= P_x^\pi\{\text{last visit of chain to } F \text{ occurs after time } n\}$$

and the last probability must go to 0 as $n \to \infty$ because the chain is transient.

Corollary. With $\bar{u}(x)$ defined for $x \in F$ by (2.8) and for $x \notin F$ by (2.9), we have for all $x \in X$,

$$\lim_{n\to\infty} E_x^\pi\{|\bar{u}(x_n) - 1|\} = 0$$

Proof. With $\phi(x)$ as defined in Lemma 2.2, we have for $x \notin F$,

$$\bar{u}(x) = \sum_{y\in F} \pi_{xy}^F \bar{u}(y) + \left(1 - \sum_{y\in F} \pi_{xy}^F\right)$$

$$= \sum_{y\in F} \pi_{xy}^F \bar{u}(y) + [1 - \phi(x)]$$

Thus,

$$E_x^\pi\{|\bar{u}(X_n)-1|\} \leqslant \sup_{y\in F} \bar{u}(y)\, E_x^\pi\{\phi(X_n)\} + E_x^\pi\{\phi(X_n)\}$$

and from the lemma, for all $x \in X$,

$$\lim_{n\to\infty} E_x^\pi\{|\bar{u}(X_n)-1|\} = 0$$

In this proof there was no loss in generality by taking $\bar{u}(x)$ as defined for $x \notin F$, since F is finite and the chain is transient.

Lemma 2.3. With $\bar{u}(x)$ defined by (2.8) for $x \in F$ and by (2.9) for $x \notin F$, we have $\bar{u}(x) = u(x)$ for all $x \in X$.

Proof. We showed in (2.15) that for *all* $x \in X$,

$$\bar{u}(x) = E_x^\pi\left\{\left[\exp \sum_{j=0}^{N-1} V(X_j)\right] \bar{u}(X_N)\right\}$$

and, because of the corollary to Lemma 2.2, the present lemma will follow if we can show that

$$\left\{\left[\exp \sum_{j=0}^{N-1} V(X_j)\right] \bar{u}(X_N)\right\}$$

are uniformly integrable. Now, by (2.14),

$$0 < \inf_{y\in X} \bar{u}(y) \leqslant \sup_{y\in X} \bar{u}(y) < \infty$$

so it suffices to show that $\{\exp \sum_{j=0}^{N-1} V(X_j)\}$ are uniformly integrable. This follows if, for some $\lambda > 1$,

$$\sup_N E_x^\pi\left\{\exp\left[\lambda \sum_{j=0}^{N-1} V(X_j)\right]\right\} \leqslant \text{const}$$

But in Lemma 2.1 we showed that if $V \in \underline{V}$, then it is in $\underline{V}_2$, i.e., $\rho(M^F) < 1$, where M^F is the matrix $\{e^{V(x)}\pi_{xy}^F,\ x, y \in F\}$. Let $M_\lambda^F = \{e^{\lambda V(x)}\pi_{xy}^F,\ x, y \in F\}$. Since the matrix is finite, its spectral radius $\rho(M_\lambda^F)$ is a continuous function of λ. Since the spectral radius is strictly less than 1 when $\lambda = 1$, there then exists $\lambda > 1$ such that $\rho(M_\lambda^F) < 1$ also. But, again by Lemma 2.1, this means for that λ, $\lambda V \in \underline{V}$, which implies

$$\sup_N E_x^\pi\left\{\exp\left[\lambda \sum_{j=0}^{N-1} V(X_j)\right]\right\} \leqslant \text{const}$$

For any $\alpha \in \underline{A}$ we introduce the Markov chain time-reversed with respect to α, i.e., for $x, y \in X$, let $\pi_{xy}(\alpha) = \pi_{yx}\alpha(y)/\alpha(x)$. The $\pi_{xy}(\alpha)$ are transition probabilities, since

$$\sum_{y \in X} \pi_{xy}(\alpha) = \frac{1}{\alpha(x)} \sum_{y \in X} \pi_{yx}\alpha(y) = \frac{\alpha(x)}{\alpha(x)} = 1$$

Since $\pi_{xy}^{(k)}(\alpha) = \pi_{yx}^{(k)}\alpha(y)/\alpha(x)$, the time-reversed chain is also irreducible. Since the original chain is transient, it has a finite Green's function, i.e., for all $x, y \in X$, $G(x, y) = \sum_{k=0}^{\infty} \pi_{xy}^{(k)} < \infty$. But the Green's function for the reversed chain $G^{\alpha}(x, y)$ is given by

$$G^{\alpha}(x, y) = \sum_{k=0}^{\infty} \pi_{xy}^{(k)}(\alpha) = \sum_{k=0}^{\infty} \pi_{yx}^{(k)} \frac{\alpha(y)}{\alpha(x)} = \frac{\alpha(y)}{\alpha(x)} G(y, x) < \infty$$

which means the reversed chain is also transient. We denote probabilities and expectations with respect to the reversed chain by $P_x^{\pi(\alpha)}\{\ \}$ and $E_x^{\pi(\alpha)}\{\ \}$, respectively.

Let $\underline{V}^{\alpha}$ be the space of functions $V\colon X \to \mathbb{R}$ that vanish outside some finite set and such that for some $x \in X$,

$$E_x^{\pi(\alpha)}\left\{\exp\left[\sum_{j=0}^{\infty} V(X_j)\right]\right\} < \infty$$

Now, in the proof of Lemma 2.1, we used only the hypotheses that the original chain was irreducible and transient. Hence, Lemma 2.1 applies also to the time-reversed chain and thus $\underline{V}^{\alpha} = \underline{V}_1^{\alpha} = \underline{V}_2^{\alpha}$.

Lemma 2.4. For any $\alpha \in \underline{A}$,

$$\underline{V} = \underline{V}^{\alpha}$$

Proof. Let $\alpha \in \underline{A}$, F be a finite subset of X, and $x, y \in F$. Then,

$$\pi_{xy}^{F}(\alpha) = \pi_{xy}(\alpha) + \sum_{z_1 \in F^c} \pi_{xz_1}(\alpha)\, \pi_{z_1 y}(\alpha) + \sum_{z_1, z_2 \in F^c} \pi_{xz_1}(\alpha)\, \pi_{z_1 z_2}(\alpha)\, \pi_{z_2 y}(\alpha) + \cdots$$

But

$$\pi_{z_1 z_2}(\alpha) = \pi_{z_2 z_1}\alpha(z_2)/\alpha(z_1)$$

so that

$$\pi^F_{xy}(\alpha) = \pi_{yx} \frac{\alpha(y)}{\alpha(x)} + \sum_{z_1 \in F^c} \pi_{yz_1} \frac{\alpha(y)}{\alpha(z_1)} \pi_{z_1 x} \frac{\alpha(z_1)}{\alpha(x)} + \cdots$$
$$= \frac{\alpha(y)}{\alpha(x)} \pi^F_{yx}$$

Hence, for a V vanishing outside F, the matrix

$$M_F(\alpha) = \{e^{V(x)}\pi^F_{xy}(\alpha), x, y \in F\} = \left\{e^{V(x)}\pi^F_{yx} \frac{\alpha(y)}{\alpha(x)}, x, y \in F\right\}$$

Since $M_F = \{e^{V(x)}\pi^F_{xy}, x, y \in F\}$, we see that if we let D be the diagonal matrix

$$D = (e^{-V(y)}\alpha(y)\, \delta_{xy}, x, y \in F)$$

then

$$M_F(\alpha) = D^{-1} M_F^T D$$

which implies $\rho(M_F(\alpha)) = \rho(M_F)$. From Lemma 2.1 and the remarks just preceding this lemma we conclude that $\underline{V} = \underline{V}^\alpha$.

Lemma 2.5. Let $V \in \underline{V}$ and define

$$u_N(x) = E^\pi_x \left\{\exp\left[\sum_{j=0}^{N-1} V(X_j)\right]\right\}$$

For $\alpha \in \underline{A}$,

$$\sup_N \sum_{x \in X} \alpha(x)\, |u_N(x) - u_{N-1}(x)| < \infty$$

Proof. It suffices to show

$$\sup_N \sum_{x \in X} \alpha(x)\, E^\pi_x \left\{\exp\left[\sum_{j=0}^{N-2} V(X_j)\right] |\exp[V(X_{N-1})] - 1| \right\} < \infty$$

and this will follow, since V has finite support, if we show that for every $y \in X$

$$\sup_N \sum_{x \in X} \alpha(x)\, E^\pi_x \left\{\left[\exp \sum_{j=0}^{N-2} V(X_j)\right] \delta_y(X_{N-1})\right\} < \infty \qquad (2.16)$$

From Lemma 2.4, $\underline{V} = \underline{V}^\alpha$, so we know

$$E_y^{\pi(\alpha)}\left\{\exp\left[\sum_{j=0}^{\infty} V(X_j)\right]\right\} < \infty$$

for every $y \in X$. Hence, for every $y \in X$,

$$\begin{aligned}
&\alpha(y) \sup_N E_y^{\pi(\alpha)}\left\{\exp\left[\sum_{j=1}^{N-1} V(X_j)\right]\right\} \\
&\quad = \alpha(y) \sup_N \sum_{x_1, x_2, \dots, x_{N-1}} \left\{\exp\left[\sum_{j=1}^{N-1} V(x_j)\right]\right\} \\
&\quad\quad \times \pi_{yx_1}(\alpha)\, \pi_{x_1x_2}(\alpha) \cdots \pi_{x_{n-2}x_{n-1}}(\alpha) < \infty
\end{aligned}$$

Since $\pi_{xy}(\alpha) = \pi_{yx}\alpha(y)/\alpha(x)$, this last becomes

$$\begin{aligned}
&\sup_N \sum_{x_1, x_2, \dots, x_{N-1}} \alpha(x_{N-1}) \left\{\exp\left[\sum_{j=1}^{N-1} V(x_j)\right]\right\} \\
&\quad \times \pi_{x_1 y} \pi_{x_2 x_1} \cdots \pi_{x_{N-1} x_{N-2}} < \infty
\end{aligned} \tag{2.17}$$

Now, if we let $x_{N-1} = x$ and $x_{N-j} = x_{j-1}$, $j = 2, 3, \dots, N-1$, (2.17) states that for $y \in X$,

$$\begin{aligned}
&\sup_N \sum_{x, x_1, x_2, \dots, x_{N-2}} \alpha(x)[\exp V(x)] \left\{\exp\left[\sum_{j=1}^{N-2} V(x_j)\right]\right\} \\
&\quad \times \pi_{xx_1} \pi_{x_1x_2} \cdots \pi_{x_{N-2}y} < \infty
\end{aligned}$$

i.e., for all $y \in X$,

$$\sup_N \sum_{x \in X} \alpha(x)\, E_x^\pi \left\{\left[\exp \sum_{j=0}^{N-2} V(X_j)\right] \delta_y(X_{N-1})\right\} < \infty$$

which is (2.16).

Lemma 2.6. $u_N(x) \to 1$ as $x \to \infty$ uniformly in N, i.e., for any $\delta > 0$ there exists a finite set F such that $x \notin F$ implies $|1 - u_N(x)| < \delta$ for all N

Proof. For any $V \in \underline{V}$,

$$E_x^\pi\left\{\left[\exp \sum_{j=0}^{N-1} V(X_j)\right] u(X_N)\right\} = u(x)$$

From Lemma 2.3 and (2.14) we then obtain

$$u_N(x) = E_x^\pi\left\{\exp\left[\sum_{j=0}^{N-1} V(X_j)\right]\right\}$$
$$\leqslant \frac{\sup_{x\in X} u(x)}{\inf_{x\in X} u(x)} \leqslant \text{const} \quad \text{for all } N \text{ and all } x$$

Since, as noted in the proof of Lemma 2.3, $V \in \underline{V}$ implies $\lambda V \in \underline{V}$ for some $\lambda > 1$, we see that

$$\sup_{\substack{x\in X\\ N}} E_x^\pi\left\{\exp\left[\lambda \sum_{j=0}^{N-1} V(X_j)\right]\right\} \leqslant \text{const}$$

for some $\lambda > 1$. This shows the uniform integrability of $\{\exp \sum_{j=0}^{N-1} V(X_j)\}$.

Now, one of our assumptions on the underlying chain is that $\lim_{x\to\infty} P_x^\pi\{\tau_F < \infty\} = 0$ for any finite subset $F \in X$. In particular, $\lim_{x\to\infty} P_x^\pi\{\tau_{\{y\}} < \infty\} = 0$, but if $y \neq x$, $P_x^\pi\{\tau_{\{y\}} < \infty\} = G(x, y)/G(y, y)$. Thus, we have $G(x, y) \to 0$ as $x \to \infty$ for every $y \in X$.

Moreover,

$$E_x^\pi\left\{\left|\sum_{j=0}^{N-1} V(X_j)\right|\right\} \leqslant E_x^\pi\left\{\sum_{j=0}^{\infty} |V(X_j)|\right\} = \sum_{y\in X} G(x, y)\, |V(y)|$$

and, since V has finite support, we see that

$$\sup_N E_x^\pi\left\{\left|\sum_{j=0}^{N-1} V(X_j)\right|\right\} \to 0 \qquad \text{as} \quad x \to \infty$$

This, together with the uniform integrability shown above, implies the lemma.

Lemma 2.7. Let $V \in \underline{V}$ and $\alpha \in \underline{A}$. Then,

$$\lim_{N\to\infty} \left|\sum_{x\in X} \alpha(x)[\log u_N(x) - \log u_{N-1}(x)] - \sum_{x\in X} \alpha(x)[u_N(x) - u_{N-1}(x)]\right| = 0 \tag{2.18a}$$

and, for any real λ,

$$\lim_{N\to\infty} \left|\sum_{x\in X} \alpha(x)\{[u_N(x)]^\lambda - [u_{N-1}(x)]^\lambda\} - \lambda \sum_{x\in X} \alpha(x)[u_N(x) - u_{N-1}(x)]\right| = 0 \tag{2.18b}$$

Proof. Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that $|1-x| < \delta$ and $|1-y| < \delta$ imply

$$|\log x - \log y - (x-y)| < \varepsilon\, |x-y|$$

For this δ, let F be the finite subset of X guaranteed by Lemma 2.6, so that $|1-u_N(x)| < \delta$ for all N provided $x \notin F$. To show the first part of (2.18), it suffices to show

$$\begin{aligned}\overline{\lim_{N\to\infty}} &\left| \sum_{x\in F} \alpha(x)[\log u_N(x) - \log u_{N-1}(x)] \right.\\ &\left. - \sum_{x\in F} \alpha(x)[u_N(x) - u_{N-1}(x)] \right|\\ &+ \overline{\lim_{N\to\infty}} \left| \sum_{x\in F^c} \alpha(x)[\log u_N(x) - \log u_{N-1}(x)] \right.\\ &\left. - \sum_{x\in F^c} \alpha(x)[u_N(x) - u_{N-1}(x)] \right| = 0 \end{aligned} \tag{2.19}$$

Since $\lim_{N\to\infty} u_N(x) = u(x) > 0$ for all $x \in X$ and since summation over $x \in F$ is just a finite sum, the first term in (2.19) is zero. For the second term in (2.19) we get, from our choice of δ and F made above,

$$\begin{aligned}\overline{\lim_{N\to\infty}} &\left| \sum_{x\in F^c} \alpha(x)[\log u_N(x) - \log u_{N-1}(x)] \right.\\ &\left. - \sum_{x\in F^c} \alpha(x)[u_N(x) - u_{N-1}(x)] \right|\\ &\leqslant \overline{\lim_{N\to\infty}}\ \varepsilon \sum_{x\in F^c} \alpha(x)\, |u_N(x) - u_{N-1}(x)|\\ &\leqslant \varepsilon \sup_N \sum_{x\in X} \alpha(x)\, |u_N(x) - u_{N-1}(x)| \end{aligned}$$

In Lemma 2.5 we showed that the multiple of ε in this last inequality is finite for any $V \in \underline{V}$ and any $\alpha \in \underline{A}$, so we have (2.19). The proof of the second statement in (2.18) is similar.

Lemma 2.8. For $V \in \underline{V}$, $\alpha \in \underline{A}$, and all $N = 1, 2,\ldots,$

$$\sum_{x\in X} \alpha(x)[(\pi u_N)(x) - u_N(x)] = 0 \tag{2.20}$$

Proof. If we can justify changing the order of summation, (2.20) follows immediately from the fact that α is an invariant measure, since

$$\sum_{x\in X} \alpha(x)[(\pi u_N)(x)-1] = \sum_{x\in X} \alpha(x)\left[\sum_{y\in X} \pi_{xy} u_N(y)-1\right]$$
$$= \sum_{x\in X} \alpha(x)[u_N(x)-1]$$

To justify the interchange, we show

$$\sum_{x\in X} \alpha(x)\,|(\pi u_N)(x)-1| < \infty$$

which is implied by

$$\sum_{x\in X} \alpha(x)\,|u_N(x)-1| < \infty \tag{2.21}$$

To show (2.21), we note that for any $x\in X$ and with $F=\operatorname{supp} V$,

$$|u_N(x)-1| = \left|E_x^\pi\left\{\exp\left[\sum_{j=0}^{N-1} V(X_j)\right]\right\}-1\right|$$
$$\leqslant [\exp(N\,\|V\|)]\,P_x^\pi\{\text{at least on } X_j\in F,\ j=0, 1,\ldots, N-1\}$$
$$= [\exp(N\,\|V\|)][\delta_x(F)+\pi^{(1)}(x, F)$$
$$+\pi^{(2)}(x, F)+\cdots+\pi^{(N-1)}(x, F)]$$

where

$$\pi^{(k)}(x, F) = \sum_{y\in F} \pi_{xy}^{(k)}, \qquad k=1, 2,\ldots, N-1$$

Thus,

$$\sum_{x\in X} \alpha(x)\,|u_N(x)-1|$$
$$\leqslant e^{N\|V\|} \sum_{x\in X} \alpha(x)[\delta_x(F)+\pi^{(1)}(x, F)+\cdots+\pi^{(N-1)}(x, F)]$$

But $\sum_{x\in X} \alpha(x)\,\delta_x(F)=\alpha(F)$ and, for each $k=1, 2,\ldots, N-1$,

$$\sum_{x\in X} \alpha(x)\,\pi^{(k)}(x, F) = \sum_{x\in X} \alpha(x) \sum_{y\in F} \pi_{xy}^{(k)} = \sum_{y\in F} \alpha(y) = \alpha(F)$$

Since F is finite and α is a σ-finite measure,

$$\sum_{x \in X} \alpha(x)\, |u_N(x) - 1| \leqslant e^{N\|V\|} N\alpha(F) < \infty$$

Although we have just shown that for each N, $\sum_{x \in X} \alpha(x)\, |u_N(x) - 1| < \infty$, in fact $\sum_{x \in X} \alpha(x)\, |u(x) - 1|$ may be infinite.

Lemma 2.9. Let $V \in \underline{V}$ and $\alpha \in \underline{A}$. Then,

$$\lim_{N \to \infty} \sum_{x \in X} \alpha(x)[u_N(x) - u_{N-1}(x)] = \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u(x)$$

Proof. First of all,

$$\begin{aligned}
u_N(x) &= E_x^\pi \left\{ \exp\left[\sum_{j=0}^{N-1} V(X_j) \right] \right\} \\
&= [\exp V(x)]\, E_x^\pi \left\{ \exp\left[\sum_{j=1}^{N-1} V(X_j) \right] \right\} \\
&= [\exp V(x)] \sum_{y \in X} \pi_{xy} E_y^\pi \left\{ \exp\left[\sum_{j=0}^{N-2} V(X_j) \right] \right\} \\
&= [\exp V(x)] \sum_{y \in X} \pi_{xy} u_{N-1}(y) \\
&= [\exp V(x)](\pi u_{N-1})(x)
\end{aligned}$$

and therefore

$$\begin{aligned}
&\lim_{N \to \infty} \sum_{x \in X} \alpha(x)[u_N(x) - u_{N-1}(x)] \\
&\quad = \lim_{N \to \infty} \sum_{x \in X} \alpha(x)[e^{V(x)}(\pi u_{N-1})(x) - u_{N-1}(x)] \\
&\quad = \lim_{N \to \infty} \left\{ \sum_{x \in X} \alpha(x)(e^{V(x)} - 1)(\pi u_{N-1})(x) \right. \\
&\qquad \left. + \sum_{x \in X} \alpha(x)[(\pi u_{N-1})(x) - u_{N-1}(x)] \right\} \\
&\quad = \lim_{N \to \infty} \left\{ \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u_N(x) \right. \\
&\qquad \left. + \sum_{x \in X} \alpha(x)[(\pi u_{N-1})(x) - u_{N-1}(x)] \right\}
\end{aligned}$$

The first term in this last expression is only a finite sum, since V vanishes outside a finite set, and so in this first term we can bring the limit inside the summation. Moreover, the second term goes to 0 as $N \to \infty$, by Lemma 2.8. Hence, since $V \in \underline{V}$,

$$\lim_{N \to \infty} \sum_{x \in X} \alpha(x)[u_N(x) - u_{N-1}(x)] = \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u(x)$$

Theorem 2.10. For $V \in \underline{V}$ and $\alpha \in \underline{A}$,

$$\lim_{N \to \infty} \frac{1}{N} \log E^{\underline{P}_n} \left\{ \exp \left[\sum_{j=0}^{N-1} \sum_{x \in X} n_j(x)\, V(x) \right] \right\}$$

$$= \sum_{x \in X} \alpha(x) \{1 - \exp[-V(x)]\}\, u(x) \tag{2.22}$$

for almost all $n(\cdot) \in Z$ (P_α-measure).

Remark. Although Theorem 2.10 is stated and proved in a form in which the null set depends on $V \in \underline{V}$, by an appropriate choice of a countable subset of $\underline{V}$ and by using standard arguments one can obtain the existence of a single null set that works simultaneously for all $V \in \underline{V}$.

Proof. From (1.1),

$$E^{\underline{P}_n} \left\{ \exp \left[\sum_{j=0}^{N-1} \sum_{x \in X} n_j(x)\, V(x) \right] \right\} = \prod_{x \in X} \left(E_x^{\pi} \left\{ \exp \left[\sum_{j=0}^{N-1} V(X_j) \right] \right\} \right)^{n(x)}$$

and therefore

$$\frac{1}{N} \log E^{\underline{P}_n} \left\{ \exp \left[\sum_{j=0}^{N-1} \sum_{x \in X} n_j(x)\, V(x) \right] \right\} = \frac{1}{N} \sum_{x \in X} n(x) \log E_x^{\pi} \left\{ \exp \left[\sum_{j=0}^{N-1} V(X_j) \right] \right\} = \frac{1}{N} \sum_{x \in X} n(x) \log u_N(x)$$

To show that (2.22) holds for almost all $n(\cdot) \in Z$ (P_α-measure) means then to show

$$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in X} n(x) \log u_N(x) = \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u(x)$$

for almost all $n(\cdot) \in Z$ (P_α-measure). For this it suffices to show that for all real λ

$$\lim_{N \to \infty} \frac{1}{N} \log E^{P_\alpha} \left\{ \exp \left[\lambda \sum_{x \in X} n(x) \log u_N(x) \right] \right\}$$
$$= \lambda \sum_{x \in X} \alpha(x) \{ 1 - \exp[-V(x)] \} \, u(x)$$

But

$$\frac{1}{N} \log E^{P_\alpha} \left\{ \exp \left[\lambda \sum_{x \in X} n(x) \log u_N(x) \right] \right\}$$
$$= \frac{1}{N} \log \prod_{x \in X} \exp(\alpha(x) \{ \exp[\lambda \log u_N(x)] - 1 \})$$
$$= \frac{1}{N} \sum_{x \in X} \alpha(x) \{ [u_N(x)]^\lambda - 1 \}$$

and so we must show

$$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in X} \alpha(x) \{ [u_N(x)]^\lambda - 1 \}$$
$$= \lambda \sum_{x \in X} \alpha(x)(1 - e^{-V(x)}) \, u(x) \tag{2.23}$$

From the usual Caesaro argument, (2.23) is implied by

$$\lim_{N \to \infty} \sum_{x \in X} \alpha(x) \{ [u_N(x)]^\lambda - [u_{N-1}(x)]^\lambda \}$$
$$= \lambda \sum_{x \in X} \alpha(x)(1 - e^{-V(x)}) \, u(x) \tag{2.24}$$

which follows from Lemmas 2.7 and 2.9.

Lemma 2.11. If $C \subset \underline{M}$ is closed and

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \log Q_{n,N}(C)$$
$$\leqslant -\sup_{V \in \underline{V}} \inf_{\sigma \in C} \left[\sum_{x \in X} \sigma(x) \, V(x) - \sum_{x \in X} \alpha(x)(1 - e^{-V(x)}) \, u(x) \right]$$

then

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{n,N}(C)$$

$$\leqslant -\inf_{\sigma\in C} \sup_{V\in \underline{V}} \left[\sum_{x\in X} \sigma(x) V(x) - \sum_{x\in X} \alpha(x)(1-e^{-V(x)}) u(x) \right]$$

$$= -\inf_{\sigma\in C} I_\alpha(\sigma)$$

Proof. First of all, one can obtain the desired result if C is compact, using methods developed in our earlier papers (cf. Ref. 1, for example).

We now describe how one can go from the result for compact sets to closed sets.

We begin by picking a $V>0$ such that

$$\sup_x \sum_y G(x, y) V(y) = \theta < 1$$

and

$$\sum_{x\in X} \alpha(x) V(x) \leqslant \text{const} < \infty$$

With such a V we then have by Portenko's lemma (see Lemma 3.1 below) that $u(x) = u_V(x) \leqslant 1/(1-\theta)$. This implies

$$\lim_{N\to\infty} \frac{1}{N} \log E^{P_\alpha} E^{P_n} \left\{ \exp\left[\sum_{j=0}^{N-1} \sum_{x\in X} V(x) n_j(x) \right] \right\}$$

$$= \lim_{N\to\infty} \sum_{x\in X} \alpha(x)[u_N(x) - u_{N-1}(x)]$$

$$= \sum_{x\in X} \alpha(x)(1-e^{-V(x)}) u(x)$$

$$\leqslant \frac{1}{1-\theta} \cdot \text{const} = K < \infty \tag{2.25}$$

Using a Chebycheff argument, we obtain from (2.25)

$$P_\alpha \left\{ n(\cdot): Q_{n,N} \left\{ \frac{1}{N} \sum_{j=0}^{N-1} \sum_{x\in X} V(x) n_j(x) \geqslant L \right\} \geqslant e^{-\beta N} \right\} \leqslant e^{KN+\beta N - LN} \tag{2.26}$$

Now, (2.26) implies

$$\overline{\lim_{L\to\infty}}\ \overline{\lim_{N\to\infty}} \frac{1}{N} \log Q_{n,N} \left\{ \frac{1}{N} \sum_{j=0}^{N-1} \sum_{x\in X} V(x) n_j(x) \geqslant L \right\} = -\infty \tag{2.27}$$

and (2.27), together with the fact that $\{\sigma: \sum_{x\in X} V(x)\sigma(x) \leqslant L\}$ is a compact set in the vague topology, implies the lemma.

3. PROPERTIES OF THE *I*-FUNCTION

In this section we prove various properties of $I_\alpha(\sigma)$ given by (1.5). Some of these results are necessary for establishing the lower bound (1.3) and will be used in the next section, where the lower bound is proved. Other properties of $I_\alpha(\sigma)$ proved here are of separate interest.

As noted earlier, the Green's function $G(x, y) < \infty$ for all $x, y \in X$, since the $\{\pi_{xy}\}$ chain is transient. If $y \neq x$, $P_x^\pi\{\tau_{\{y\}} < \infty\} = G(x, y)/G(y, y)$, and therefore our hypothesis on $P_x^\pi\{\tau_F < \infty\}$ implies

$$\begin{aligned} &G(x, y) \to 0 \quad \text{as} \quad y \to \infty \quad \text{for any } x \in X \\ &G(x, y) \to 0 \quad \text{as} \quad x \to \infty \quad \text{for any } y \in X \end{aligned} \tag{3.1}$$

We also recall that we are assuming

$$\sup_{x \in X} G(x, x) < \infty \tag{3.2}$$

Both (3.1) and (3.2) will be used in this section.

Lemma 3.1. Let $V: X \to R$ vanish outside some finite subset of X and be such that

$$\sup_{x \in X} \sum_{y \in X} G(x, y) \, |V(y)| \leqslant \theta < 1 \tag{3.3}$$

Then, $V \in \underline{V}$ and $|1 - u_V(x)| \leqslant \theta/(1-\theta)$ for each $x \in X$.

Proof. Let $r \geqslant 1$ and consider

$$\begin{aligned} &E_x^\pi\{[V(X_0) + V(X_1) + \cdots]^r\} \\ &\quad \leqslant E_x^\pi\{[|V(X_0)| + |V(X_1)| + \cdots]^r\} \\ &\quad = E_x^\pi\left\{\sum_{i_1, i_2, \ldots, i_r} |V(X_{i_1})| \, |V(X_{i_2})| \cdots |V(X_{i_r})|\right\} \\ &\quad \leqslant r! \, E_x^\pi\left\{\sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r} |V(X_{i_1})| \, |V(X_{i_2})| \cdots |V(X_{i_r})|\right\} \\ &\quad \leqslant r! \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{r-1}} E_x^\pi\left\{|V(X_{i_1})| \, |V(X_{i_2})|\right. \\ &\qquad \times \cdots |V(X_{i_{r-1}})| \left. E_{X_{i_{r-1}}}^\pi\left(\sum_{j=0}^\infty |V(X_j)|\right)\right\} \\ &\quad \leqslant \theta r! \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{r-1}} E_x^\pi\{|V(X_{i_1})| \, |V(X_{i_2})| \cdots |V(X_{i_{r-1}})| \end{aligned}$$

the last inequality following from (3.3). If we keep repeating this process, we end up with

$$E_x^\pi\{[V(X_0)+V(X_1)+\cdots]^r\} \leqslant \theta^r r!$$

and thus

$$|1-u_V(x)| = \left|1-E_x^\pi\left\{\exp\left[\sum_{j=0}^{\infty} V(X_j)\right]\right\}\right| \leqslant \sum_{r=1}^{\infty} \frac{\theta^r r!}{r!} = \frac{\theta}{1-\theta}$$

which implies $u_V(x)<\infty$, so that $V \in \underline{V}$.

Lemma 3.2. Let $x_1, x_2,..., x_N$ be a finite subset of X for which $G(x_i, x_i) \leqslant C$, $i=1, 2,..., N$, $G(x_i, x_j) \leqslant \varepsilon$ for all $i \neq j$, $i, j=1, 2,..., N$. Let $\gamma>0$ be so small that $\gamma[C+(N-1)\varepsilon]=\theta<1$. Let $\alpha \in \underline{A}$. For any $\sigma \in \underline{M}$ such that $I_\alpha(\sigma)<l$, we have

$$\begin{aligned} &\frac{e^\gamma-1}{\gamma}\left[\sum_{i=1}^{N} \alpha(x_i)\right]\left(1-\frac{\theta}{1-\theta}\right)-\frac{l}{\gamma} \\ &\leqslant \sum_{i=1}^{N} \sigma(x_i) \\ &\leqslant \frac{1-e^{-\gamma}}{\gamma}\left[\sum_{i=1}^{N} \alpha(x_i)\right]\left(1+\frac{\theta}{1-\theta}\right)+\frac{l}{\gamma} \end{aligned} \tag{3.4}$$

Proof. Let

$$V(x)=\begin{cases}\gamma & \text{if } \ x=x_i, \quad i=1, 2,..., N \\ 0 & \text{otherwise}\end{cases}$$

Then, using the maximum principle,

$$\begin{aligned} &\sup_{x \in X} \sum_{y \in X} G(x, y)\, V(y) \\ &\quad= \sup_{1 \leqslant i \leqslant N} \sum_{y \in X} G(x_i, y)\, V(y) \\ &\quad= \gamma \sup_{1 \leqslant i \leqslant N} \sum_{j=1}^{N} G(x_i, x_j) \\ &\quad\leqslant \gamma[C+(N-1)\varepsilon]=\theta<1 \end{aligned}$$

Hence, from Lemma 3.1, for the V just selected, $u_V(x) \leqslant 1+\theta/(1-\theta)$, and $V \in \underline{V}$. Since $\sigma \in \underline{M}$ is such that $I_\alpha(\sigma) \leqslant l$, we then have for this V,

$$\sum_{x \in X} \sigma(x)\, V(x) - \sum_{x \in X} \alpha(x)(1-e^{-V(x)})\, u_V(x) \leqslant l$$

i.e.,

$$\gamma \sum_{i=1}^{N} \sigma(x_i) - (1-e^{-\gamma})\left[\sum_{i=1}^{N} \alpha(x_i)\right]\left(1+\frac{\theta}{1-\theta}\right) \leqslant l$$

or

$$\sum_{i=1}^{N} \sigma(x_i) \leqslant \frac{1-e^{-\gamma}}{\gamma}\left[\sum_{i=1}^{N} \alpha(x_i)\right]\left(1+\frac{\theta}{1-\theta}\right)+\frac{l}{\gamma}$$

If we choose instead

$$V(x)=\begin{cases} -\gamma, & x=x_i, \quad i=1,2,\ldots,N \\ 0, & \text{otherwise} \end{cases}$$

then again Lemma 3.1 implies this $V \in \underline{V}$ also and $u_V(x) \geqslant 1-\theta/(1-\theta)$, which, by the same argument as above, leads to the inequality on the left of (3.4).

Theorem 3.3. Let $\alpha \in \underline{A}$ and $\sigma \in \underline{M}$ such that $I_\alpha(\sigma)=l<\infty$. Then:

1. If, for some sequence $\{y_n\} \to \infty$, $\alpha(y_n) \to 0$, then $\sigma(y_n) \to 0$.
2. If, for some sequence $\{y_n\} \to \infty$, $\alpha(y_n) \to L>0$, then $\sigma(y_n) \to L$ also.
3. If, for some sequence $\{y_n\} \to \infty$, $\alpha(y_n) \to \infty$, then $\alpha(y_n)/\sigma(y_n) \to 1$.

These three statements combine into: If $\{y_n\} \to \infty$, then $[1+\alpha(y_n)]/[1+\sigma(y_n)] \to 1$.

Proof. We prove each of the three statements. First assume for some sequence $\{y_n\} \to \infty$, $\alpha(y_n) \to 0$. To prove statement 1, we assume the contrary, i.e., there exists $\delta>0$ such that $\sigma(y_n) \geqslant \delta$ for all n. From (3.2) let $\sup_{x \in X} G(x,x) < C$ and let $\varepsilon>0$ be given. Let γ_0 be so small that $\gamma<\gamma_0$ implies

$$\frac{1-e^{-\gamma}}{\gamma}\left(1+\frac{\gamma(C+1)}{1-\gamma(C+1)}\right)<2$$

Choose $\gamma>0$, but $\gamma<\min(1/2(C+1), \gamma_0)$. Choose N so large that $l/N\gamma<\varepsilon/2$. Because we are assuming here that $\alpha(y_n) \to 0$ and because of (3.1) we can choose N elements $x_1, x_2,\ldots, x_N$ in X such that $G(x_i, x_j) \leqslant 1/N$ for $i \neq j$, $i, j=1,2,\ldots,N$, and such that $\alpha(x_j) \leqslant \varepsilon/4$, $j=1,2,\ldots,N$.

Now,

$$\gamma[C+(N-1)(1/N)] \leqslant \gamma[C+1]=\theta<1/2$$

and hence from Lemma 3.2,

$$N\delta \leqslant \sum_{i=1}^{N} \sigma(x_i) \leqslant \frac{1-e^{-\gamma}}{\gamma}\left(1+\frac{\gamma(C+1)}{1-\gamma(C+1)}\right) N\frac{\varepsilon}{4}+\frac{l}{\gamma} \leqslant \frac{N\varepsilon}{2}+\frac{l}{\gamma}$$

i.e.,

$$\delta \leqslant \frac{\varepsilon}{2}+\frac{l}{N\gamma} \leqslant \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we have a contradiction and statement 1 is proved.

To prove statement 2, suppose $\{y_n\} \to \infty$ such that $\alpha(y_n) \to L > 0$. Again we assume the contrary, i.e., there is a subsequence of $\{y_n\}$, which we will again label $\{y_n\}$, such that either $\sigma(y_n) \geqslant L' > L$ for all $n = 1, 2,...$, or $\sigma(y_n) \leqslant L' < L$ for all $n = 1, 2,...$. For the first case, let $\varepsilon > 0$ be given and choose $\gamma > 0$ so small that $\gamma[C+1] = \theta < 1/2$ and

$$\frac{1-e^{-\gamma}}{\gamma}\left[1+\frac{\gamma(C+1)}{1-\gamma(C+1)}\right] < 1+\varepsilon$$

Choose N so large that $l/N\gamma < \varepsilon$. For that N we can find points $x_1, x_2,..., x_N$ such that $G(x_i, x_j) \leqslant 1/N$ for all $i \neq j$, $i, j = 1, 2,..., N$, and such that $\alpha(x_j) \leqslant L + \varepsilon$. Thus, from Lemma 3.2,

$$NL' \leqslant \sum_{i=1}^{N} \sigma(x_i) \leqslant N(L+\varepsilon)(1+\varepsilon) + l/\gamma$$

or

$$L' \leqslant (L+\varepsilon)(1+\varepsilon) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, this contradicts $L' > L$. A similar argument using the lower estimate in Lemma 3.2 takes care of the second case, $L' < L$, and statement 2 is proved.

Finally, to show statement 3, let $\{y_n\} \to \infty$ be a sequence such that $\alpha(y_n) \to \infty$. Again assuming the contrary, suppose there is a subsequence $\{y_n\}$ such that $\sigma(y_n) \geqslant \lambda\alpha(y_n)$ for some $\lambda > 1$, $n = 1, 2,...$, or $\sigma(y_n) \leqslant \lambda\alpha(y_n)$ for some $\lambda < 1$, $n = 1, 2,...$. Take the first case, and use the upper estimate of Lemma 3.2 with $N = 1$. Choose $\gamma > 0$ so small that $\gamma C = \theta < 1$ and

$$\frac{1-e^{-\gamma}}{\gamma}\left(1+\frac{\gamma C}{1-\gamma C}\right) < 1+\frac{\varepsilon}{2}$$

where $\varepsilon > 0$ is given. Choose $M > 0$ so large that $l/M\gamma < \varepsilon/2$. Now, there is an $x \in X$ for which $\alpha(x) > M$. Hence from Lemma 3.2 for this particular x,

$$\frac{\sigma(x)}{\alpha(x)} \leqslant \frac{1-e^{-\gamma}}{\gamma}\left(1+\frac{\gamma C}{1-\gamma C}\right)+\frac{l}{\alpha(x)\,\gamma}$$
$$\leqslant \left(1+\frac{\varepsilon}{2}\right)+\frac{l}{M\gamma} \leqslant 1+\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this contradicts $\sigma(x)/\alpha(x) \geqslant \lambda > 1$. Similarly, of course, for the other alternative, and the proof is complete.

Lemma 3.4. Let $\alpha \in \underline{A}$ and $\beta \in \underline{A}$. Assume for some $\sigma \in \underline{M}$ that both $I_\alpha(\sigma) < \infty$ and $I_\beta(\sigma) < \infty$. Then $\alpha \equiv \beta$.

Proof. First we want to prove the following: if $\alpha \in \underline{A}$ and $\beta \in \underline{A}$ and if $h(x) = \beta(x)/\alpha(x) \to 1$ as $x \to \infty$, then $\alpha \equiv \beta$.

As in Section 2, let $\pi_{xy}(\alpha) = \pi_{yx}\alpha(y)/\alpha(x)$ and recall that $\{\pi_{xy}(\alpha)\}$ are transition probabilities for an irreducible chain. Now,

$$\sum_{y \in X} \pi_{xy}(\alpha)\, h(y) = \sum_{y \in X} \pi_{yx} \frac{\alpha(y)}{\alpha(x)} \frac{\beta(y)}{\alpha(y)}$$
$$= \frac{1}{\alpha(x)} \sum_{y \in X} \pi_{yx}\beta(y) = \frac{\beta(x)}{\alpha(x)} = h(x)$$

so that h is a harmonic function of the chain time-reversed with respect to α. By hypothesis, $h(x) \to 1$ at ∞, but, for an irreducible Markov chain, if a bounded harmonic function approaches 1 at infinity, it is then identically 1.

Now, to prove this lemma, we have from Theorem 3.3 that $\{y_n\} \to \infty$ implies $[1+\sigma(y_n)]/[1+\alpha(y_n)] \to 1$ and also $[1+\sigma(y_n)]/[1+\beta(y_n)] \to 1$, i.e., $[1+\alpha(y_n)]/[1+\beta(y_n)] \to 1$. Since, by hypothesis, $\{\pi_{xy}\}$ is doubly stochastic, we see that 1 is an invariant measure for the π-chain, i.e., $1 \in \underline{A}$. Since the sum of two invariant measures is also an invariant measure, the statement at the beginning of this proof implies $1+\alpha(\cdot) \equiv 1+\beta(\cdot)$, i.e., $\alpha(\cdot) \equiv \beta(\cdot)$.

Lemma 3.5. Let $\alpha \in \underline{A}$ and $\sigma \in \underline{M}$ such that $I_\alpha(\sigma) < \infty$. Then, for all $y \in X$,

$$\lim_{k \to \infty} \sum_{x \in X} \sigma(x)\, \pi_{xy}^{(k)} = \alpha(y) \tag{3.5}$$

Proof. As noted in the preceding lemma, since $\{\pi_{xy}\}$ is doubly stochastic, 1 is an invariant measure, and therefore to show (3.5), it suffices to show

$$\lim_{k \to \infty} \sum_{x \in X} [\sigma(x)+1]\, \pi_{xy}^{(k)} = \alpha(y)+1 \tag{3.6}$$

for all $y \in X$. Let

$$h(x) = \frac{\sigma(x)+1}{\alpha(x)+1}, \qquad \tilde{\pi}_{xy} = \pi_{yx} \frac{\alpha(y)+1}{\alpha(x)+1}$$

Now

$$\tilde{\pi}_{xy}^{(k)} = \pi_{yx}^{(k)} \frac{\alpha(y)+1}{\alpha(x)+1}$$

and

$$\begin{aligned}\sum_{x \in X} [\sigma(x)+1]\, \pi_{xy}^{(k)} &= \sum_{x \in X} \frac{\alpha(y)+1}{\alpha(x)+1} \tilde{\pi}_{yx}^{(k)} [\sigma(x)+1] \\ &= [\alpha(y)+1] \sum_{x \in X} h(x)\, \tilde{\pi}_{yx}^{(k)} \\ &= [\alpha(y)+1]\, E_y^{\tilde{\pi}}\{h(X_k)\}\end{aligned}$$

As we have argued several times already, the $\tilde{\pi}$-chain is also transient, which implies, for every $y \in X$, $X_k \to \infty$ for almost all paths starting from y ($\tilde{\pi}$-measure). Moreover, from Theorem 3.3, $h(x_k) \to 1$ if $x_k \to \infty$, so by bounded convergence $\lim_{k \to \infty} E_y^{\tilde{\pi}}\{h(X_k)\} = 1$ and we have (3.6) for all $y \in X$ and the lemma.

We now prove a succession of lemmas leading up to Theorem 3.11, which will be used in Section 4 in order to prove the lower bound (1.3).

In these lemmas as well as in Theorem 3.11, F will denote a fixed, finite subset of X having elements $x_1, x_2, \ldots, x_k$. Let $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_k\}$ be a point in R^k and let $C = \{V: X \to \mathbb{R}\}$. Let $T: R^k \to C$ be the mapping given by

$$\begin{aligned} V(x_i) &= \theta_i, & i &= 1, 2, \ldots, k \\ V(x) &= 0, & x &\neq x_i \end{aligned}$$

Let $\underline{Q} \in R^k$ consist of those $\boldsymbol{\theta} \in R^k$ such that $T\boldsymbol{\theta} \in \underline{V}$. For $\alpha \in \underline{A}$ and $\boldsymbol{\theta} \in \underline{Q}$, define

$$\Phi(\boldsymbol{\theta}) = \Phi_F(\boldsymbol{\theta}) = \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-\theta_i})\, u(\boldsymbol{\theta}; x_i) - \sum_{i=1}^{k} \alpha(x_i)\, \theta_i$$

where, of course,

$$u(\boldsymbol{\theta}; x_i) = u_V(x_i) = E_{x_i}^{\pi} \left\{ \exp\left[\sum_{j=0}^{\infty} V(X_j) \right] \right\}$$

with $V = T\boldsymbol{\theta}$.

In the first of these lemmas we use the notation of Section 2. Recall that $\pi^F_{xy} = P^\pi_x\{X_\tau = y; \tau < \infty\}$, where $\tau = \inf_{j \geq 1}\{X_j \in F\}$, and

$$\eta_x = 1 - \sum_{y \in F} \pi^F_{xy} = P^\pi_x\{X_j \notin F, j \geq 1\}$$

For the time-reversed chain with respect to an $\alpha \in \underline{A}$, i.e., the chain with transition probabilities $\pi_{xy}(\alpha) = \pi_{yx}\alpha(y)/\alpha(x)$, we use the notations

$$\pi^F_{xy}(\alpha) = P^{\pi(\alpha)}_x\{X_\tau = y; \tau < \infty\}$$

$$\eta_x(\alpha) = 1 - \sum_{y \in F} \pi^F_{xy}(\alpha) = P^{\pi(\alpha)}_x\{X_j \notin F, j \geq 1\}$$

as before.

Lemma 3.6. For $\alpha \in \underline{A}$ and $\boldsymbol{\theta} \in \underline{Q}$,

$$\Phi_F(\boldsymbol{\theta}) = \sum_{i=1}^k \alpha(x_i)\, \eta_{x_i}(\alpha)\, u(\boldsymbol{\theta}; x_i) - \sum_{i=1}^k \alpha(x_i)\, \eta_{x_i} - \sum_{i=1}^k \alpha(x_i)\, \theta_i \tag{3.7}$$

Proof. Since $\boldsymbol{\theta} \in \underline{Q}$, $V \in \underline{V}$ and therefore from (2.6) we have for any $x \in X$,

$$u_V(x) = e^{V(x)}\left[\sum_{y \in F} \pi^F_{xy} u_V(y) + \eta_x\right]$$

Thus,

$$\begin{aligned}
&\sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u_V(x) \\
&\quad= \sum_{x \in F} \alpha(x)(1 - e^{-V(x)})\, u_V(x) \\
&\quad= \sum_{x \in F} \alpha(x)\, u_V(x) - \sum_{x \in F} \alpha(x)\left[\sum_{y \in F} \pi^F_{xy} u_V(y) + \eta_x\right] \\
&\quad= \sum_{y \in F}\left[\alpha(y) - \sum_{x \in F} \alpha(x)\, \pi^F_{xy}\right] u_V(y) - \sum_{x \in F} \alpha(x)\, \eta_x
\end{aligned} \tag{3.8}$$

For x, $y \in F$,

$$\begin{aligned}
\pi^F_{xy} &= \pi_{xy} + \sum_{z_1 \notin F} \pi_{xz_1}\pi_{z_1y} + \sum_{\substack{z_1 \notin F \\ z_2 \notin F}} \pi_{xz_1}\pi_{z_1z_2}\pi_{z_2y} + \cdots \\
&= \pi^{(1)}_{xy} + \pi^{(2)}_{xy} + \cdots + \pi^{(n)}_{xy} + \cdots
\end{aligned}$$

and therefore

$$\begin{aligned}
\alpha(y) - \sum_{x\in F} \alpha(x)\,\pi^F_{xy} \\
&= \sum_{x\in X} \alpha(x)\,\pi_{xy} - \sum_{x\in F} \alpha(x)\,\pi^F_{xy} \\
&= \sum_{x\notin F} \alpha(x)\,\pi_{xy} - \sum_{x\in F} \alpha(x)[\pi^F_{xy} - \pi_{xy}] \\
&= \sum_{x\notin F} \alpha(x)\,\pi_{xy} - \sum_{x\in F} \alpha(x) \sum_{n=2}^{\infty} \pi^{(n)}_{xy} \\
&= \sum_{x\notin F} \sum_{z_1\in X} \alpha(z_1)\,\pi_{z_1x}\pi_{xy} - \sum_{x\in F} \alpha(x) \sum_{n=2}^{\infty} \pi^{(n)}_{xy} \\
&= \sum_{x\notin F} \sum_{z_1\notin F} \alpha(z_1)\,\pi_{z_1x}\pi_{xy} \\
&\quad - \sum_{x\in F} \alpha(x) \left[\sum_{n=2}^{\infty} \pi^{(n)}_{xy} - \sum_{z_1\notin F} \pi_{xz_1}\pi_{z_1y} \right] \\
&= \sum_{x\notin F} \sum_{z_1\notin F} \alpha(z_1)\,\pi_{z_1x}\pi_{xy} - \sum_{x\in F} \alpha(x) \sum_{n=3}^{\infty} \pi^{(n)}_{xy}
\end{aligned}$$

If we repeat this process we arrive at the following expression, which holds for any $n = 1, 2,...$:

$$\begin{aligned}
\alpha(y) - \sum_{x\in F} \alpha(x)\,\pi^F_{xy} \\
&= \sum_{z_1, z_2,..., z_n \notin F} \alpha(z_1)\,\pi_{z_1z_2}\pi_{z_2z_3}\cdots\pi_{z_ny} \\
&\quad - \sum_{x\in F} \alpha(x) \sum_{k=n+1}^{\infty} \pi^{(k)}_{xy}
\end{aligned} \tag{3.9}$$

Letting $n \to \infty$ in (3.9), we see that (cf. proof of Lemma 2.3)

$$\begin{aligned}
\alpha(y) - \sum_{x\in F} \alpha(x)\,\pi^F_{xy} \\
&= \lim_{n\to\infty} \sum_{z_1, z_2,..., z_n \notin F} \alpha(z_1)\,\pi_{z_1z_2}\pi_{z_2z_3}\cdots\pi_{z_ny} \\
&= \lim_{n\to\infty} \sum_{z_1, z_2, z_3,..., z_n \notin F} \alpha(y)\,\pi_{yz_1}(\alpha)\,\pi_{z_1z_2}(\alpha)\cdots\pi_{z_{n-1}z_n}(\alpha) \\
&= \alpha(y)\,\eta_y(\alpha)
\end{aligned} \tag{3.10}$$

Using (3.10) in (3.8), we get

$$\sum_{x \in X} \alpha(x)(1-e^{-V(x)})\, u_V(x) = \sum_{y \in F} \alpha(y)\, \eta_y(\alpha)\, u_V(y) - \sum_{x \in F} \alpha(x)\, \eta_x$$

Therefore,

$$\Phi_F(\boldsymbol{\theta}) = \sum_{i=1}^{k} \alpha(x_i)\, \eta_{x_i}(\alpha)\, u(\boldsymbol{\theta}; x_i) - \sum_{i=1}^{k} \alpha(x_i)\, \eta_{x_i} - \sum_{i=1}^{k} \alpha(x_i)\, \theta_i$$

which is the lemma.

Lemma 3.7. $\underline{O}$ is an open, convex set in R^k, $\Phi(\boldsymbol{\theta}) \geqslant 0$ for $\boldsymbol{\theta} \in \underline{O}$, and if $\tilde{\boldsymbol{\theta}} \in R^k$ such that $\tilde{\boldsymbol{\theta}} \in \partial \underline{O}$ and $\boldsymbol{\theta}^{(n)} \to \tilde{\boldsymbol{\theta}}$ as $n \to \infty$, then $\Phi(\boldsymbol{\theta}^{(n)}) \to \infty$.

Proof. In Section 2 we noted that if $V \in \underline{V}$, there exists $\lambda > 1$ such that $\lambda V \in \underline{V}$, which implies $\underline{O}$ is open. From Lemma 2.1, $\underline{V} = \underline{V}_2$, so $V \in \underline{V}$ implies $\rho(M_F) < 1$, where M_F is the matrix $\{e^{V(x)} \pi^F_{xy},\ x,\ y \in F\}$. Since here F is fixed, $\rho(M_F) = \rho(\boldsymbol{\theta})$ and $\boldsymbol{\theta} \in \underline{O}$ implies $\rho(\boldsymbol{\theta}) < 1$, i.e., the convex function $\log \rho(\boldsymbol{\theta}) < 0$ for $\boldsymbol{\theta} \in \underline{O}$, which implies $\underline{O}$ is convex. That $\Phi(\boldsymbol{\theta}) \geqslant 0$ for $\boldsymbol{\theta} \in \underline{O}$ follows by applying Jensen's inequality to the definition of $\Phi(\boldsymbol{\theta})$.

Although $\underline{O}$ is open and convex, it is not bounded, and indeed we later must examine the behavior of $\Phi(\boldsymbol{\theta})$ as $\boldsymbol{\theta}$ stays in $\underline{O}$ but approaches ∞. Here the sequence $\{\boldsymbol{\theta}^{(n)}\} \in \underline{O}$ converges to the finite point $\tilde{\boldsymbol{\theta}} \in \partial \underline{O}$. In the definition of $\Phi(\boldsymbol{\theta})$, the term

$$\sum_{i=1}^{k} \alpha(x_i)(1-e^{-\theta_i})\, u(\boldsymbol{\theta}; x_i)$$

is troublesome because, depending on the signs of the θ_i, some terms are positive and some negative, presenting possible cancellations. This problem is obviated by the decomposition in Lemma 3.6. Thus,

$$\Phi(\boldsymbol{\theta}^{(n)}) = \sum_{i=1}^{k} \alpha(x_i)\, \eta_{x_i}(\alpha)\, u(\boldsymbol{\theta}^{(n)}; x_i) - \sum_{i=1}^{k} \alpha(x_i)\, \eta_{x_i} - \sum_{i=1}^{k} \alpha(x_i)\, \theta_i^{(n)} \tag{3.11}$$

In (3.11) not all of the factors $\eta_{x_i}(\alpha)$, $i = 1, 2,..., k$, can be zero, because the chain time-reversed with respect to α is transient. Since $\underline{O}$ is open, $\tilde{\boldsymbol{\theta}} \notin \underline{O}$, and hence $u(\tilde{\boldsymbol{\theta}}; x_i) = \infty$, $i = 1, 2,..., k$. From Fatou's lemma, we see then that

$u(\boldsymbol{\theta}^{(n)}; x_i) \to \infty$ as $\boldsymbol{\theta}^{(n)} \to \tilde{\boldsymbol{\theta}}$ for $i = 1, 2, \ldots, k$. Thus, the first term in (3.11) becomes infinite as $\boldsymbol{\theta}^{(n)} \to \tilde{\boldsymbol{\theta}}$ and so $\Phi(\boldsymbol{\theta}^{(n)}) \to \infty$ as $\boldsymbol{\theta}^{(n)} \to \tilde{\boldsymbol{\theta}}$, completing the lemma.

Now we must examine the behavior of $\Phi(\boldsymbol{\theta})$ as $\boldsymbol{\theta} \to \infty$. There exist rays along which $\Phi(\boldsymbol{\theta})$ remains bounded and we have to take these into account in what follows. To this end, we define a point $\mathbf{a} \in R^k$ to be *special* if $\sup_{\lambda > 0} \Phi(\lambda \mathbf{a}) \leqslant \text{const}$. Let $S = \{\mathbf{a} \in R^k, \mathbf{a} \text{ special}\}$. Clearly, $S \subset \underline{O}$.

Lemma 3.8. If $\mathbf{a} \in S$, then $-\mathbf{a} \in S$ and $\Phi(\mathbf{a}) = 0$. S is a linear subspace of R^k and if $\mathbf{b} \in \underline{O}$ and $\mathbf{a} \in S$, then $\mathbf{b} + \mathbf{a} \in \underline{O}$ and $\Phi(\mathbf{b} + \mathbf{a}) = \Phi(\mathbf{b})$.

Proof. From the definition of $\Phi(\boldsymbol{\theta})$ we see that it is convex and $\Phi(\mathbf{0}) = 0$. Thus, if $\Phi(\lambda \mathbf{a}) \leqslant c$ for all $\lambda > 0$, then $\Phi(\lambda \mathbf{a}) \equiv 0$ for all $\lambda > 0$. Since Φ is analytic in $\boldsymbol{\theta}$, we have $\Phi(\lambda \mathbf{a}) \equiv 0$ for all real λ. To show that S is a linear subspace, it suffices to show that $\mathbf{a}_1 \in S$ and $\mathbf{a}_2 \in S$ imply $\frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2) \in S$. But

$$0 \leqslant \Phi\left(\frac{\mathbf{a}_1 + \mathbf{a}_2}{2}\right) \leqslant \frac{1}{2}\left[\Phi(\mathbf{a}_1) + \Phi(\mathbf{a}_2)\right] = 0$$

so not only is S a linear subspace, but $\Phi(\boldsymbol{\theta}) \equiv 0$ for $\boldsymbol{\theta} \in S$.

Let $\mathbf{b} \in \underline{O}$ and $\mathbf{a} \in S$. Since $\underline{O}$ is open, there exists $\varepsilon > 0$ small enough so that $[1/(1-\varepsilon)]\, \mathbf{b} \in \underline{O}$. By definition, $(1/\varepsilon)\, \mathbf{a} \in S \subset \underline{O}$ and since $\underline{O}$ is a convex set,

$$\mathbf{b} + \mathbf{a} = (1-\varepsilon)\frac{1}{1-\varepsilon}\mathbf{b} + \varepsilon\left(\frac{1}{\varepsilon}\right)\mathbf{a} \in \underline{O}$$

By the convexity of Φ and since $\Phi(\mathbf{a}) = 0$, we have

$$\Phi(\mathbf{a} + \mathbf{b}) \leqslant (1-\varepsilon)\, \Phi\left(\frac{\mathbf{b}}{1-\varepsilon}\right)$$

Convex functions being continuous, we obtain from this that

$$\Phi(\mathbf{a} + \mathbf{b}) \leqslant \Phi(\mathbf{b}) \qquad \text{for all} \quad \mathbf{a} \in S \text{ and } \mathbf{b} \in \underline{O} \tag{3.12}$$

In (3.12) we can replace $\mathbf{a}$ by $-\mathbf{a}$, giving $\Phi(\mathbf{b} - \mathbf{a}) \leqslant \Phi(\mathbf{b})$ for all $\mathbf{a} \in S$ and $\mathbf{b} \in \underline{O}$. But we just showed $\mathbf{a} + \mathbf{b} \in \underline{O}$ and therefore this last inequality gives $\Phi(\mathbf{b}) \leqslant \Phi(\mathbf{a} + \mathbf{b})$, which, together with (3.12), finishes the lemma.

We have noted that $S \subset \underline{O}$. Let $\underline{O}_0 = \{\boldsymbol{\theta}' \in \underline{O}, \boldsymbol{\theta}' \perp S\}$, so $\underline{O}_0$ is a closed subset of $\underline{O}$, and any $\boldsymbol{\theta} \in \underline{O}$ can be written $\boldsymbol{\theta} = \boldsymbol{\theta}' + \mathbf{a}$, where $\boldsymbol{\theta}' \in \underline{O}_0$ and $\mathbf{a} \in S$.

Lemma 3.9. Let $\{\boldsymbol{\theta}^{(n)}\} \in \underline{O}_0$ such that $\|\boldsymbol{\theta}^{(n)}\| \to \infty$. Then, $\Phi(\boldsymbol{\theta}^{(n)}) \to \infty$.

Proof. Assume the contrary; then, there exists a constant C such that $\Phi(\boldsymbol{\theta}^{(n)}) \leqslant C$ for all n. For $\lambda > 0$,

$$\begin{aligned}\Phi\left(\frac{\lambda\boldsymbol{\theta}^{(n)}}{\|\boldsymbol{\theta}^{(n)}\|}\right) &= \Phi\left(\frac{\lambda}{\|\boldsymbol{\theta}^{(n)}\|}\boldsymbol{\theta}^{(n)} + \left(1 - \frac{\lambda}{\|\boldsymbol{\theta}^{(n)}\|}\right)\cdot \mathbf{0}\right)\\ &\leqslant \frac{\lambda}{\|\boldsymbol{\theta}^{(n)}\|}\Phi(\boldsymbol{\theta}^{(n)})\\ &\leqslant \frac{C\lambda}{\|\boldsymbol{\theta}^{(n)}\|}\end{aligned}$$

which implies $\Phi(\lambda\boldsymbol{\theta}^{(n)}/\|\boldsymbol{\lambda}^{(n)}\|) \to 0$ as $n \to \infty$. Since $\underline{Q}_0$ is closed and, for each n, $\|\boldsymbol{\theta}^{(n)}/\|\boldsymbol{\theta}^{(n)}\|\,\| = 1$, there exists a subsequence, which we will label $\boldsymbol{\theta}^{(n)}/\|\boldsymbol{\theta}^{(n)}\|$, converging to some element $\mathbf{a} \in \underline{Q}_0$ such that $\|\mathbf{a}\| = 1$. Since Φ is continuous, we have for all $\lambda > 0$, $\Phi(\lambda\mathbf{a}) = 0$, which means $\mathbf{a}$ is special. But $\mathbf{a} \in \underline{Q}_0$ means $\mathbf{a} \perp \mathbf{a}$, which is impossible, since $\mathbf{a}$ is a unit vector.

Lemma 3.10. Let $\alpha \in \underline{A}$ and $\sigma \in \underline{M}$ such that $I_\alpha(\sigma) < \infty$. If $\mathbf{a} \in S$, then

$$\sum_{i=1}^{k} \sigma(x_i)\, a_i = \sum_{i=1}^{k} \alpha(x_i)\, a_i$$

Proof.

$$\begin{aligned}I_\alpha(\sigma) &= \sup_{V \in \underline{V}}\left[\sum_{x \in X} \sigma(x)\, V(x) - \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u_V(x)\right]\\ &\geqslant \sup_{\boldsymbol{\theta} \in \underline{Q}}\left[\sum_{i=1}^{k} \sigma(x_i)\, \theta_i - \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-\theta_i})\, u(\boldsymbol{\theta}; x_i)\right]\end{aligned}$$

since the first supremum is over V's vanishing outside any finite subset of X, whereas the second supremum is just over V's vanishing outside our given finite subset F.

Thus, using the fact that $\Phi(\mathbf{a}) = 0$ for $\mathbf{a} \in S$,

$$\begin{aligned}I_\alpha(\sigma) &\geqslant \sup_{\boldsymbol{\theta} \in \underline{Q}}\left[\sum_{i=1}^{k} \sigma(x_i)\, \theta_i - \sum_{i=1}^{k} \alpha(x_i)\, \theta_i - \Phi_F(\boldsymbol{\theta})\right]\\ &\geqslant \sup_{\mathbf{a} \in S}\left[\sum_{i=1}^{k} \sigma(x_i)\, a_i - \sum_{i=1}^{k} \alpha(x_i)\, a_i - \Phi_F(\mathbf{a})\right]\\ &= \sup_{\mathbf{a} \in S}\left[\sum_{i=1}^{k} \sigma(x_i)\, a_i - \sum_{i=1}^{k} \alpha(x_i)\, a_i\right]\end{aligned}$$

Since S is a linear subspace, this last supremum is either ∞ or 0. The former is ruled out by our hypothesis $I_\alpha(\sigma) < \infty$, and hence we have this lemma, because

$$\sup_{\mathbf{a} \in S} \sum_{i=1}^{k} [\sigma(x_i) - \alpha(x_i)]\, a_i = 0$$

implies

$$\sum_{i=1}^{k} [\sigma(x_i) - \alpha(x_i)]\, a_i = 0 \qquad \text{identically in } S$$

Theorem 3.11. Let $\alpha \in \underline{A}$ and $\sigma \in \underline{M}$ such that $I_\alpha(\sigma) < \infty$. Let $\sigma' = (1-\varepsilon)\,\sigma + \varepsilon\alpha$ for $\varepsilon > 0$ fixed, and let F be a finite subset of X. In the variational problem

$$\sup_{\boldsymbol{\theta} \in \underline{Q}} \left[\sum_{i=1}^{k} \sigma'(x_i)\, \theta_i - \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-\theta_i})\, u(\boldsymbol{\theta}; x_i) \right] \tag{3.13}$$

the supremum is attained at a point $\boldsymbol{\theta}^* \in \underline{Q}$.

Proof. Since $\alpha \in \underline{A}$, $I_\alpha(\alpha) = 0$, and the convexity of the I-function implies

$$I_\alpha(\sigma') \geqslant \sup_{\boldsymbol{\theta} \in \underline{Q}} \left[\sum_{i=1}^{k} \sigma'(x_i)\, \theta_i - \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-\theta_i})\, u(\boldsymbol{\theta}; x_i) \right]$$

we see that the supremum in (3.13) is finite. Also, since $\mathbf{0} \in \underline{Q}$, we see that the supremum in (3.13) is nonnegative.

With $\Phi_F(\boldsymbol{\theta})$ as defined earlier, we can write (3.13) as

$$\sup_{\boldsymbol{\theta} \in \underline{Q}} \left[\sum_{i=1}^{k} \sigma'(x_i)\, \theta_i - \sum_{i=1}^{k} \alpha(x_i)\, \theta_i - \Phi_F(\boldsymbol{\theta}) \right] \tag{3.14}$$

As noted before, for any $\boldsymbol{\theta} \in \underline{Q}$ we can write $\boldsymbol{\theta} = \mathbf{a} + \mathbf{b}$, where $\mathbf{a} \in S$ and $\mathbf{b} \in \underline{Q}_0 = \{\boldsymbol{\theta}' \in \underline{Q}\colon\ \boldsymbol{\theta}' \perp S\}$. From Lemma 3.8, $\Phi(\mathbf{a} + \mathbf{b}) = \Phi(\mathbf{b})$. Since $I_\alpha(\sigma') < \infty$, for this $\mathbf{a} \in S$ we have from Lemma 3.10 that $\sum_{i=1}^{k} \sigma'(x_i)\, a_i = \sum_{i=1}^{k} \alpha(x_i)\, a_i$. Thus, (3.14) becomes

$$\sup_{\mathbf{b} \in \underline{Q}_0} \left[\sum_{i=1}^{k} \sigma'(x_i)\, b_i - \sum_{i=1}^{k} \alpha(x_i)\, b_i - \Phi_F(\mathbf{b}) \right]$$

$$= \sup_{\mathbf{b} \in \underline{Q}_0} \left[(1-\varepsilon) \sum_{i=1}^{k} \sigma(x_i)\, b_i \right.$$

$$\left. - (1-\varepsilon) \sum_{i=1}^{k} \alpha(x_i)\, b_i - \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-b_i})\, u(\mathbf{b}; x_i) + \sum_{i=1}^{k} \alpha(x_i)\, b_i \right]$$

$$= \sup_{\mathbf{b} \in \underline{Q}_0} \left\{ (1-\varepsilon) \left[\sum_{i=1}^{k} \sigma(x_i)\, b_i - \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-b_i})\, u(\mathbf{b}; x_i) \right] - \varepsilon \Phi_F(\mathbf{b}) \right\} \tag{3.15}$$

Let $\mathbf{b}^{(n)}$ be a maximizing sequence in $\underline{Q}_0$. There are three possibilities: (1) $\|\mathbf{b}^{(n)}\| \to \infty$, (2) $\mathbf{b}^{(n)} \to \mathbf{b} \in \partial \underline{Q}_0$, or (3) $\mathbf{b}^{(n)} \to \mathbf{\theta}^*$, a point in $\underline{Q}_0 \subset \underline{Q}$.

Assume possibility 1 first. By hypothesis,

$$\infty > I_\alpha(\sigma) \geqslant \sup_{\mathbf{b} \in \underline{Q}_0} \left[\sum_{i=1}^{k} \sigma(x_i)\, b_i - \sum_{i=1}^{k} \alpha(x_i)(1 - e^{-b_i})\, u(\mathbf{b}; x_i) \right]$$

so that in the last line of (3.15) the quantity in square brackets is bounded, whereas Lemma 3.9 gives us $\Phi_F(\mathbf{b}^{(n)}) \to \infty$. Hence, in the last line of (3.15) the quantity in curly braces goes to $-\infty$ as $\|\mathbf{b}^{(n)}\| \to \infty$, which contradicts the supremum in (3.13) being nonnegative.

Next, assume possibility 2 holds. From Lemma 3.7, $\Phi_F(\mathbf{b}^{(n)}) \to \infty$ if $\mathbf{b}^{(n)} \to \mathbf{b} \in \partial \underline{Q}_0 \subset \partial \underline{Q}$, and, just as above, the quantity in square brackets in the last line of (3.15) remains bounded. Thus, the quantity in curly braces in the last line of (3.15) again goes to $-\infty$ as $\mathbf{b}^{(n)} \to \mathbf{b} \in \partial \underline{Q}_0$, which gives the same contradiction, and the proof is complete.

4. THE LOWER BOUND

Let $\alpha \in \underline{A}$. To prove the lower bound (1.3), it suffices, since G is open, to show that for any $\sigma \in \underline{M}$ such that $I_\alpha(\sigma) < \infty$,

$$\varliminf_{N \to \infty} \frac{1}{N} \log Q_{n,N}(N_\sigma) \geqslant -I_\alpha(\sigma) \tag{4.1}$$

for almost all $n(\cdot) \in Z$ (P_α-measure), where N_σ is any $\underline{M}$-neighborhood of σ.

We can choose $\varepsilon > 0$ so small that $\sigma' = (1-\varepsilon)\,\sigma + \varepsilon\alpha$ is an element of N_σ and then we can choose an $\underline{M}$-neighborhood of σ', call it $N_{\sigma'}$, such that $N_{\sigma'} \subset N_\sigma$. Moreover, since the I-function is convex and $I_\alpha(\alpha) = 0$, we see that

$$I_\alpha(\sigma') \leqslant (1-\varepsilon)\, I_\alpha(\sigma) + \varepsilon I_\alpha(\alpha) \leqslant I_\alpha(\sigma)$$

Thus, to show (4.1), it suffices to show for σ' of the form above and any $\underline{M}$-neighborhood $N_{\sigma'}$ of σ' that

$$\varliminf_{N \to \infty} \frac{1}{N} \log Q_{n,N}(N_{\sigma'}) \geqslant -I_\alpha(\sigma') \tag{4.2}$$

for almost all $n(\cdot) \in Z$ (P_α-measure).

Sets in M of the form

$$B_{\sigma'} = \{\lambda(\cdot) \in \underline{M} \colon |\lambda(x) - \sigma'(x)| < \delta \text{ for all } x \in F, \text{ a finite subset of } X)$$

form a basis for the weak topology in $\underline{M}$, so that, to show (4.2), it suffices to show, for any set $B_{\sigma'}$ in the basis and $\sigma' = (1-\varepsilon)\,\sigma + \varepsilon\alpha$, that

$$\varliminf_{N\to\infty} \frac{1}{N} \log Q_{n,N}(B_{\sigma'}) \geqslant -I_\alpha(\sigma') \tag{4.3}$$

for almost all $n(\cdot) \in Z$ (P_α-measure).

With this in mind, we let F be a finite subset of X having elements $x_1, x_2, ..., x_k$. Let $\underline{Q} \subset R^k$ be the set of $\boldsymbol{\theta} \in R^k$ having the property that $V: X \to \mathbb{R}$ defined by

$$V(x) = \begin{cases} \theta_i & \text{if} \quad x = x_i, \quad i = 1, 2, ..., k \\ 0 & \text{otherwise} \end{cases}$$

is in $\underline{V}$. As noted in Section 3, $\underline{Q}$ is an open set in R^k. For each $N = 1, 2, ...,$ define a probability measure μ_N on R^k by: if $A \subset R^k$,

$$\mu_N(A) = \underline{P}_n\{(L_N(x_1), L_N(x_2), ..., L_N(x_k)) \in A\}$$

Let

$$M_N(\boldsymbol{\theta}) = \int \exp(N\langle \boldsymbol{\theta}, \mathbf{y}\rangle)\, d\mu_N(\mathbf{y}) = E^{\underline{P}_n}\left\{\exp\left[N \sum_{i=1}^k \theta_i L_N(x_i)\right]\right\}$$

which exists if $\boldsymbol{\theta} \in \underline{Q}$. What is important is that, if $\alpha \in \underline{A}$ and $\boldsymbol{\theta} \in \underline{Q}$, we have from Lemma 2.10 that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \log M_N(\boldsymbol{\theta}) \\
&\quad = \lim_{N\to\infty} \frac{1}{N} \log E^{P_n}\left\{\exp\left[N \sum_{i=1}^k \theta_i L_N(x_i)\right]\right\} \\
&\quad = \lim_{N\to\infty} \frac{1}{N} \log E^{P_n}\left\{\exp\left[\sum_{j=0}^{N-1} \sum_{i=1}^k \theta_i n_j(x_i)\right]\right\} \\
&\quad = \lim_{N\to\infty} \frac{1}{N} \log E^{P_n}\left\{\exp\left[\sum_{j=0}^{N-1} \sum_{x\in X} n_j(x)\, V(x)\right]\right\} \\
&\quad = \sum_{x\in X} \alpha(x)\{1 - \exp[-V(x)\}\, u_V(x) \\
&\quad = \sum_{i=1}^k \alpha(x_i)[1 - \exp(-\theta_i)]\, u(\boldsymbol{\theta}; x_i) = \psi(\boldsymbol{\theta})
\end{aligned}$$

for almost all $n(\cdot) \in Z$ (P_α-measure).

We note that $\psi(\boldsymbol{\theta})$ is continuously differentiable on $\underline{O}$. For $\alpha \in \underline{A}$, $\sigma \in \underline{M}$ such that $I_\alpha(\sigma) < \infty$, and with $\sigma' = (1-\varepsilon)\,\sigma + \varepsilon\alpha$, we proved (Theorem 3.11) that in the variational problem,

$$\sup_{\boldsymbol{\theta} \in \underline{O}} \left[\sum_{i=1}^{k} \sigma'(x_i)\,\theta_i - \psi(\boldsymbol{\theta}) \right]$$

the supremum is actually attained at a point $\boldsymbol{\theta}^* \in \underline{O}$. This, of course, implies $(\nabla\psi)(\boldsymbol{\theta}^*) = \boldsymbol{\sigma}'$, where $\boldsymbol{\sigma}' = (\sigma'(x_i),\ i = 1, 2,..., k)$.

All of these observations become important because of the following lemma, which is a standard, general result, the proof of which can be found, for example, in Ref. 10.

Lemma 4.1. Let $\{\mu_N\}$ be a family of probability measures on R^k and assume

$$M_N(\boldsymbol{\theta}) = \int \exp(N\langle \boldsymbol{\theta}, \mathbf{y} \rangle)\, d\mu_N(\mathbf{y})$$

exists for $\boldsymbol{\theta}$ belonging to an open set $O \subset R^k$. Assume also that

$$\lim_{N \to \infty} \frac{1}{N} \log M_N(\boldsymbol{\theta} = \psi(\boldsymbol{\theta})$$

exists for $\boldsymbol{\theta} \in O$ and that $\psi(\boldsymbol{\theta})$ is continuously differentiable in O. Finally, assume $\boldsymbol{\sigma}' \in R^k$ is such that $(\nabla\psi)(\boldsymbol{\theta}^*) = \boldsymbol{\sigma}'$ for some $\boldsymbol{\theta}^* \in O$. Then,

$$\varliminf_{N \to \infty} \frac{1}{N} \log \mu_N(N_{\boldsymbol{\sigma}'}) \geqslant -[\langle \boldsymbol{\sigma}', \boldsymbol{\theta}^* \rangle - \psi(\boldsymbol{\theta}^*)] \tag{4.4}$$

where $N_{\boldsymbol{\sigma}'}$ is an R^k neighborhood of $\boldsymbol{\sigma}'$.

Since, as we just noted, all the hypotheses of Lemma 4.1 are satisfied, we conclude that

$$\begin{aligned}
&\varliminf_{N \to \infty} \frac{1}{N} \log \mu_N(N_{\boldsymbol{\sigma}'}) \\
&\quad = \varliminf_{N \to \infty} \frac{1}{N} \log P_n\{(L_N(x_1), L_N(x_2),..., L_N(x_k)) \in N_{\boldsymbol{\sigma}'}) \\
&\quad = \varliminf_{N \to \infty} \frac{1}{N} \log Q_{n,N}(N_{\boldsymbol{\sigma}'}) \\
&\quad \geqslant -[\langle \boldsymbol{\sigma}', \boldsymbol{\theta}^* \rangle - \psi(\boldsymbol{\theta}^*)] \\
&\quad = -\sup_{\boldsymbol{\theta} \in \underline{O}} \left[\sum_{i=1}^{k} \sigma'(x_i)\,\theta_i - \psi(\boldsymbol{\theta}) \right]
\end{aligned}$$

for almost all $n(\cdot) \in Z$ (P_α-measure). But

$$I_\alpha(\sigma') = \sup_{V \in \underline{V}} \left[\sum_{x \in X} \sigma'(x)\, V(x) - \sum_{x \in X} \alpha(x)(1 - e^{-V(x)})\, u_V(x) \right]$$
$$= \sup_{\boldsymbol{\theta} \in \underline{Q}} \left[\sum_{i=1}^{k} \sigma'(x_i)\, \theta_i - \psi(\boldsymbol{\theta}) \right] \tag{4.5}$$

because the first supremum is over V's vanishing outside any finite subset of X, whereas the second supremum is just over V's vanishing outside a fixed, finite set F. From (4.4) and (4.5) we have

$$\varliminf_{N \to \infty} \frac{1}{N} \log Q_{n,N}(N_{\sigma'}) \geqslant -I_\alpha(\sigma') \tag{4.6}$$

for almost all $n(\cdot) \in Z$ (P_α-measure).

Now, since F was any finite subset of X, and since $N_{\sigma'}$ is any R^k neighborhood of σ', we see that (4.6) implies (4.3) and hence the lower bound (1.3).

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